TMA06 TNs 202xJ

TMA 01

Solution to Question 6 – 7 marks

We must show that d satisfies (M1)–(M3) of
Definition 2.1 of Chapter 14, HB page 72.

(M1) Certainly $d(i,j) \geq 0$ for any $i, j \in \mathbb{N}$. Moreover $d(i,j) = 0$ if and only if

$$\left|\frac{i}{i+1} - \frac{j}{j+1}\right| = 0,$$

which holds if and only if

$$\frac{i}{i+1} = \frac{j}{j+1}$$

if and only if

$$i(j+1) = j(i+1)$$

— that is, if and only if $i = j$. Hence (M1) holds.

(M2) For each $i, j \in \mathbb{N}$,

$$d(i,j) = \left|\frac{i}{i+1} - \frac{j}{j+1}\right| = \left|\frac{j}{j+1} - \frac{i}{i+1}\right| = d(j,i).$$

Hence (M2) holds.

(M3) Let $i, j, k \in \mathbb{N}$. Then, on using the triangle inequality for the modulus function,

$$\begin{aligned} d(i,k) &= \left|\frac{i}{i+1} - \frac{k}{k+1}\right| \\ &= \left|\left(\frac{i}{i+1} - \frac{j}{j+1}\right) + \left(\frac{j}{j+1} - \frac{k}{k+1}\right)\right| \\ &\leq \left|\frac{i}{i+1} - \frac{j}{j+1}\right| + \left|\frac{j}{j+1} - \frac{k}{k+1}\right| = d(i,j) + d(j,k). \end{aligned}$$

Hence (M3) holds.

We conclude that d is a metric for $\mathbb{N}$.

An alternative (and faster) solution is to show that d is the pull-back of the restriction to $\mathbb{N}$ of the Euclidean metric by the map $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = \frac{n}{n+1}$
Theorem 1.2 of Chapter 15, HB page 74, and
Worked Exercise 1.3 of Chapter 15, page 117.

Solution to Question 7 – 13 marks

Note that there are many possible ways to justify the answers to (a) part (ii) and (b). The following are examples of possible solutions and contain more detail than is necessary. The strategy on ***page 83 of the handbook*** lists some examples of possible approaches to these questions.

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(a) (i) A is both d_A-open and d_A-closed since any set is both open and closed as a metric subspace of itself ***Lemma 2.2 of Chapter 16, HB page 81***.

(ii) A is not d-open since (for example) $B_d(\frac{1}{2}, \epsilon)$ contains $\min\{\{\frac{3}{4}, \frac{1}{2} + \frac{\epsilon}{2}\} \notin A$ ***Theorem 2.9 of Chapter 16, HB page 82***. A is not d-closed since (a_n) where $a_n = \frac{1}{p}$ where p is the n^{th} prime is a null sequence but $0 \notin A$.
An alternative way to do this question is to show that the complement of A in $\mathbb{R}$ is not d-open by, for example, remarking that for any $r > 0$ we can find a prime p such that $\frac{1}{p} < r$ and so $\frac{1}{p} \in B_d(0, r) \cap A$. Therefore there is no d-open ball around the point 0 in $\mathbb{R} - A$.
Theorem 2.9 of Chapter 16, HB page 82 and
Strategy, HB page 83.

(iii) $\text{Int}_{(\mathbb{R},d)}(A) = \emptyset$ (by an argument similar to that in ***Exercise 4.4 of Chapter 16***).
$\text{Cl}_{(\mathbb{R},d)}(A) = A \cup \{0\}$ (since 0 is the only d-closure point of A that is not in A) ***Definition 3.6, Worked Exercise 3.7*** and
Theorem 3.8 of Chapter 16, HB page 84.

(b) (i) Closed and not open.
This set consists of the union of 201 images of the closed interval $[-1, 1]$ under the maps $f_i : [0, 1] \to \mathbb{R}^2$ taking x to (x, i), for $-100 \leq i \leq 100$ and $-1 \leq x \leq 1$.
There are many ways to see that this is d-closed
Strategy, HB page 83.
For example the note in the text after Worked Exercise 1.9 of chapter 16 states that any set of the form $[a, b] \times [c, d]$ is $d^{(2)}$-closed. This shows that the set consisting of the line segment $[-1, 1] \times [n, n]$ is $d^{(2)}$-closed and X is a finite union of sets of this form. Alternatively the complement is d-open. To see this, it can be noted that for any $(x, y) \in \mathbb{R}^2 - X$ let $r = \min(\{|y - z|, z \in \mathbb{Z}\} \cup \{||x| - 1|\})$. Then $B_d((x, y), r) \subseteq \mathbb{R}^2 - X$. However it is sufficient to simply say that there is a d-open ball around every point in the complement that is contained within the complement.
X is not $d^{(2)}$-open because (for example) any open ball around the point $(0, 0)$ intersects the complement of X. (Specifically $(0, \frac{r}{2} \in B_d((0, 0), r) \cap (\mathbb{R}^2 - X)$ but details are not necessary).

(ii) Closed and not open. This set consists of the union of countably many copies of the closed interval $[-1, 1]$.
There are many ways to see that this is d-closed.
One example is to note that the complement is open. To see this let (a, b) be any point in the half plane $A^+ = \{(x, y) \in R^2 : x > 1\}$. Then let $r = \frac{|a-1|}{2}$ and note that $B_b((a, b), r) \subseteq A$ and so A is d-open. Similarly the half plane $A^- = \{(x, y) \in R^2 : x < -1\}$ is d-open. Let $U_z = \bigcup_{i \in [-1,1]} B_d((i, z + \frac{1}{2}), \frac{1}{2})$ and note that $\mathbb{R}^2 - X$ is the union of A^+, A^- and U_z for $z \in \mathbb{Z}$.
$X$is not $d^{(2)}$-open because (for example) any open ball around the point $(0, 0)$ intersects the complement of X.

(iii) Neither open nor closed. By the density property of the reals, any $B_d((0, q), r)$ with $q \in \mathbb{Q}$ must contain a point of the form $(0, z)$ where $z \notin \mathbb{Q}$ so X is not open and any $B_d((0, q), r)$ with $q \notin \mathbb{Q}$ must contain a point of the form $(0, z)$ where $z \in \mathbb{Q}$ so $\mathbb{R}^2 - X$ is not open.

Solution to Question 8 – 8 marks

We use the Arzelà-Ascoli Theorem, which is
Theorem 3.12 of Chapter 22, HB p131, and show that A is $d_{\max}$-closed, pointwise bounded and equicontinuous from which it follows that A is $d_{\max}$-sequentially compact.

Claim: A is $d_{\max}$-closed.

Proof of claim Suppose that (f_n) is a sequence of functions from A that $d_{\max}$-converges to $f \in C[0,1]$. We need to show that $f \in A$.

Thus we need to show that both $|f(x)| \leq \exp(x)$ for each $x \in [0,1]$ and $|f(x) - f(y)| \leq \sqrt{|x-y|}$ for each $x, y \in [0,1]$.

For each $x \in [0,1]$, we can use the Triangle Inequality (for modulus) to estimate that for each $n \in \mathbb{N}$.

$$\begin{aligned} |f(x)| &= |f(x) - f_n(x) + f_n(x)| \\ &\leq |f(x) - f_n(x)| + |f_n(x)| \\ &\leq d_{\max}(f_n, f) + \exp(x). \end{aligned}$$

But $(d_{\max}(f_n, f))$ is a (real) null sequence and n is arbitrary, hence $|f(x)| \leq \exp(x)$ for each $x \in [0,1]$.

Similarly for each $x, y \in [0,1]$, we can use the Triangle Inequality (for modulus) twice to estimate that for each $n \in \mathbb{N}$.

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq 2d_{\max}(f_n, f) + |f_n(x) - f_n(y)| \\ &\leq 2d_{\max}(f_n, f) + \sqrt{|x-y|}. \end{aligned}$$

But $(d_{\max}(f_n, f))$ is a (real) null sequence and n is arbitrary, hence $|f(x) - f(y)| \leq \sqrt{|x-y|}$ for each $x, y \in [0,1]$.

We conclude that A is $d_{\max}$-closed.

Claim: A is pointwise bounded.

Proof of claim

This follows from the result of ***Exercise 3.5 of Chapter 22*** and ***HB p130, Examples of pointwise bounded sets, 2***.

Claim: A is equicontinuous Proof of claim

Let $\epsilon > 0$ be given, then taking $\delta = \epsilon^2$ gives $\sqrt{r} < \epsilon$, whenever $0 \leq r < \delta$. In particular, if $|x - y| < \delta$, then for each $f \in A$,

$$|f(x) - f(y)| \leq \sqrt{|x-y|} < \epsilon$$

and so A is an equicontinuous set of functions.

Solution to Question 9 – 10 marks

(a) (i)

$$d(\mathbf{a}_{10,i}, \mathbf{a}_{9,i}) = \sum_{i=1}^{\infty} |a_{10,i} - a_{9,i}| = \sum_{i=1}^{10} |a_{10,i} - a_{9,1}| = |a_{10,10}| = p^{-10}.$$

(ii) For $n \in \mathbb{N}$,

$$d(\mathbf{a}_{n+1,i}, \mathbf{a}_{n,i}) = \sum_{i=1}^{\infty} |a_{n+1,i} - a_{n,i}| = \sum_{i=1}^{n+1} |a_{n+1,i} - a_{n,i}| = \frac{1}{p^{n+1}}$$

.

(iii) For $m, n \in \mathbb{N}$ with $m > n$, we use the Triangle inequality for d a total of $m - n - 2$ times to estimate

$$\begin{aligned} d(\mathbf{a}_m, \mathbf{a}_n) &\leq d(\mathbf{a}_m, \mathbf{a}_{m-1}) + d(\mathbf{a}_{m-1}, \mathbf{a}_{m-2}) + \cdots + d(\mathbf{a}_{n+1}, \mathbf{a}_n) \\ &= \textstyle\sum_{i=n+1}^{m} \frac{1}{p^i}, \text{ by part (ii)} \\ &\leq \frac{1}{p^n}. \end{aligned}$$

Hence for $\epsilon > 0$ if we choose $N \in \mathbb{N}$ so that $1/p^N < \epsilon$, then for $m, n > N$ we have $d(\mathbf{a}_m, \mathbf{a}_n) \leq \frac{1}{p^N} < \epsilon$.

Hence $(\mathbf{a}_n)$ is a Cauchy sequence for d.

(b) If the sequence $(\mathbf{a}_n)$ from part (a) converged to $\mathbf{b} = (b_n) \in \ell_0$, then there would be $N \in \mathbb{N}$ so that $b_n = 0$ for $n > N$,

but then for $n > N$, $d(\mathbf{a}_n, \mathbf{b}) \geq p^{-N+1} > 0$.

Hence, despite being a Cauchy sequence, $(\mathbf{a}_n)$ does not converge in ℓ_0. We conclude that ℓ_0 is not complete for d.

Chapter 23. See Exercises 1.2, 1.3 and 1.4 and the proof of Theorem 2.7.

Solution to Question 10 – 12 marks

(a) (i)

$$\begin{aligned} B_{d_G}(g,1) &= \{h \in G : d(g,h) < 1\} \\ &= \{h \in G : d(g,h) = 0\} \\ &= \{h \in G : gh^{-1} \in \{e\}\} \\ &= \{g\}. \end{aligned}$$

$$\begin{aligned} B_{d_G}[g,1] &= \{h \in G : d(g,h) \leq 1\} \\ &= \{h \in G : d(g,h) = 0\} \cup \{h \in G : d(g,h) = 1\} \\ &= \{g\} \cup \{h : gh^{-1} \in H_1\} \\ &= H_1 g, \text{ (since if } gh^{-1} \in H_1, \\ &\qquad \text{then } (gh^{-1})^{-1} = hg^{-1} \in H_1 \text{ so } h \in H_1 g.) \end{aligned}$$

$S_{d_G}(g,1) = H_1 g - \{g\}$

(ii)

$$\begin{aligned} B_{d_G}(g,2) &= \{h \in G : d(g,h) < 2\} \\ &= \{h \in G : d(g,h) = 0 \text{ or } 1\} \\ &= H_1 g. \end{aligned}$$

$$\begin{aligned} B_{d_G}[g,2] &= \{h \in G : d(g,h) \leq 2\} \\ &= \{h \in G : d(g,h) = 0\} \cup \{h \in G : d(g,h) = 1\} \cup \{h \in G : d(g,h) = 2\} \\ &= H_2 g. \end{aligned}$$

$S_{d_G}(g,1) = H_2 g - H_1 g.$

(b) By ***Theorem 2.9 of Chapter 16, HB p82*** a set, $A \subseteq G$ is d_G-open if, and only if for each $a \in A$ we can find an $r > 0$ such that $B_{g_A}(a,r) \subseteq A$. Since $B_{d_G}(g,1) = \{g\}$ for any $g \in G$ we have that $\{g\}$ is open for any $g \in G$. Therefore any set $A = \bigcup_{a \in A} \{a\}$ is d_G-open. Since any set is the complement of an open set we have that all sets are closed.for argument

(c) (i) G is d_G-connected if, and only if G is the trivial group consisting of one element (since if there is any non-identity element $g \in G$ then the set $\{g\}$ is a proper non-empty subset of G that is both d_G-open and d_G-closed and so G is d_G-disconnected by ***Theorem 3.2 of Chapter 21, HB p120***).

(ii) G is d_G-compact if, and only if G is finite. (Any finite set is compact ***Examples, HB p132*** and if G is infinite then $\mathcal{S} = \{\{g\} : g \in G\}$ is an infinite open cover with no finite subcover.)

(iii) Any group is d_G-complete (since taking $\epsilon = \frac{1}{2}$ in ***Definition 1.3 of Chapter 23, HB p135*** shows that the only d_G-Cauchy sequences are eventually constant and so convergent).